On a class of adjoint functional equations

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To Béla Szőkefalvi-Nagy on his sixtieth birthday

1. Problem. Let A and B denote *vector mean values* in R^m, this term to be defined below. Consider the two *adjoint functional equations*

$$f[A(s,t)] = B[f(s), f(t)], \tag{1.1}$$

$$g[B(a,b)] = A[g(a), g(b)]. \tag{1.2}$$

A study of these equations is proposed involving intrinsic properties of the solutions, classification of the latter with continuous solutions as one type and boundary affine solutions as the other, spread of continuity, and construction of solutions by a process of successive interpolation. The equations are *inverses* of each other in the sense that there exist solutions f_0 and g_0 such that

$$f_0[g_0(a)] = a, \quad g_0[f_0(s)] = s \tag{1.3}$$

in the domains of definition of g_0 and f_0.

In the theory of vector meanvalues the case

$$B(a,b) = \tfrac{1}{2}(a+b) \tag{1.4}$$

is basic and it plays an important role also in this paper where we are trying to extend the results obtained in this special case (see HILLE [2]) to the more general one. Some additional material for the special case is to be found in Section 11 below.

2. On vector meanvalues. In [2] the author based the discussion of vector meanvalues on a system of postulates analogous to those of A. N. KOLMOGOROV and M. NAGUMOV in the linear case.

The discussion involves various consequences of the natural partial ordering of R^m. We write

$$x = (x_1, x_2, \dots, x_m), \quad y = (y_1, y_2, \dots, x_m)$$

*) Research supported in part by NSF Grant GU-2582.

and define

(2.1) $$x \leqq y$$

iff

(2.2) $$x_j \leqq y_j, \qquad j = 1, 2, \dots, m.$$

Should "$<$" hold here for all j, we write $x<y$. For a finite set of vectors x_p in R^m, distinct or not, we write

(2.3) $$u = (u_1, u_2, \dots, u_m) = \inf V$$

for the vector whose jth coordinate is the infimum of the jth coordinates of the vectors x_p of V. Similarly we define

(2.4) $$v = (v_1, v_2, \dots, v_m) = \sup V.$$

The set of all vectors $x=(x_1, x_2, \dots, x_m)$ such that

(2.5) $$u_j \leqq x_j \leqq v_j, \qquad j = 1, 2, \dots, m,$$

is called the *closed cellular hull* $C[V]$ of V. If in all the inequalities where $u_j<v_j$ we replace "$\leqq$" by "$<$" we obtain a subset $C^0[V]$ of $C[V]$ known as the *open cellular hull* of $C[V]$. This is not necessarily an open set in the topology of R^m, but, unless all the x_p's are equal, there is a subspace of lower dimension in which $C^0[V]$ is open. These concepts are due to J. B. Miller [3] and have been further explored by C. T. Ng [4, 5, 6].

The vectors to be admitted in forming meanvalues will be restricted to an open convex set G in R^m. G is supposed to contain the closed pyramid Π defined by

(2.6) $$x_j \geqq 0, \quad j = 1, 2, \dots, m, \quad x_1 + x_2 + \dots + x_m \leqq 1.$$

We can now formulate the postulates:

M_1. For each finite set V of vectors $x_1, x_2, \dots, x_n$ in G, not necessarily distinct, there exists a meanvalue $M(V) = M(x_1, x_2, \dots, x_n)$, a vector in G.

M_2. M is a symmetric continuous function, strictly increasing in each of its arguments.

M_{21}. $M(V) \subset C^0[V]$.

M_{22}. For t fixed and s variable, both in G, the mapping $t \to M(s, t)$ is open, injective and continuous, uniformly with respect to t on compact subsets of G.

M_3. $M(x, x, \dots, x) = x$.

M_4. Let $1<k<n$ and set $M(x_1, x_2, \dots, x_k) = y$. Then

$$M(x_1, \dots, x_k, x_{k+1}, \dots, x_n) = M(y, \dots, y, x_{k+1}, \dots, x_n)$$

where y is repeated k times.

The meanvalues A and B in equations (1. 1) and (1. 2) are supposed to satisfy these conditions. We write D for G in the first case, E in the second; they may be distinct or identical.

3. Some properties of meanvalues. As consequences of M_3 and M_4 we note

Lemma 1. *The meanvalue of k copies of the set V is the same as of one copy*

$$M(kV) = M(V). \tag{3.1}$$

Lemma 2. *We have*

$$M[M(V_1), M(V_2)] = M(V_1 \cup V_2). \tag{3.2}$$

Here it should be noted that each vector figures with its proper multiplicity: if x occurs k_1 times in V_1 and k_2 times in V_2, then it figures k_1+k_2 times in $V_1 \cup V_2$.

Lemma 3. *From a set V of n vectors, select k vectors, $k<n$, and form the meanvalue. Repeat this for the $N = \binom{n}{k}$ choices of k objects from a set of n. Let V_k be the set of the corresponding N meanvalues. Then*

$$M(V_k) = M(V). \tag{3.3}$$

Proof. We consider an enlarged set V^* made up of

$$\binom{n-1}{k-1}$$

copies of V. By Lemma 1, $M(V^*)=M(V)$. Here the set V^* has kN vectors which may be arranged into N sets of k vectors each. This is to be done in such a manner that the elements in the pth set are precisely those k vectors chosen in the pth selection. Let y_p be the meanvalue of these vectors. In forming $M(V^*)$ we can replace each of the vectors in the pth set by y_p using M_4. We repeat this for each p so that

$$M(V^*) = M(y_1, \ldots, y_1, \ldots, y_N, \ldots, y_N)$$

where each y_j figures k times. Using Lemma 1 again we can contract the last expression to $M(y_1, \ldots, y_N)=M(V_k)$ as asserted.

Corollary. *For each $k<n$*

$$M(V) \subset C^0[V_k]. \tag{3.4}$$

A simple argument shows that

$$C^0[V] \supset C^0[V_2] \supset \cdots \supset C^0[V_k] \tag{3.5}$$

which expresses that meanvalues are *variation-reducing*.

Lemma 4. *We have*

$$\|M(s,t)-\tfrac{1}{2}(s+t)\| \leqq \tfrac{1}{2}\sqrt{3}\,\|s-t\|. \tag{3.6}$$

Proof. The coordinates of the vectors M, s and t satisfy the inequalities

$$\inf(s_j,t_j)-s_j \leqq M_j-s_j \leqq \sup(s_j,t_j)-s_j$$

for $j=1,2,\ldots,m$. This implies that

$$|M_j-s_j| \leqq |t_j-s_j| \tag{3.7}$$

for all j so that

$$\|M(s,t)-s\| \leqq \|s-t\| \tag{3.8}$$

and by symmetry

$$\|M(s,t)-t\| \leqq \|s-t\|. \tag{3.9}$$

The geometric meaning of these two inequalities is that $M(s,t)$ lies in the domain common to two spheres, one with center at s, the other at t having the same radius $\|s-t\|$. The inequality (3. 6) follows. Here equality can hold iff $s=t$ in which case, of course, equality holds for all j in (3. 7).

The main result of [2] can be expressed as follows.

Theorem M. *The equation*

$$h[M(s,t)] = \tfrac{1}{2}[h(s)+h(t)] \tag{3.10}$$

with initial conditions

$$h(0)=0, \quad h(u_1)=u_1, \ldots, h(u_m)=u_m \tag{3.11}$$

where the u's are the unit vectors, $u_j=(\delta_{jk})$, has a unique solution h which is one-to-one. In terms of this solution

$$h[M(s_1,s,\ldots,s_n)] = \frac{1}{n}\sum_{1}^{n} h(s_j) \tag{3.12}$$

for all n and all s_j in G.

This representation leads to further properties of the meanvalues. It is clear that

$$M(s,u) \neq M(s,v) \text{ if } u \neq v. \tag{3.13}$$

A more general form of this inequality is

Lemma 5. *If $u=M(u_1,\ldots,u_p)$, $v=M(v_1,\ldots,v_p)$ and $u\neq v$, then*

$$M(s_1,\ldots,s_k,u_1,\ldots,u_p) \neq M(s_1,\ldots,s_k,v_1,\ldots,v_p). \tag{3.14}$$

Proof. We use the representation of M given by (3. 12). Thus

$$(k+p)h[M(s_1, \ldots, u_p)] = \sum_1^k h(s_j) + \sum_1^p h(u_n) = \sum_1^k h(s_j) + ph(u) \neq$$

$$\neq \sum_1^k h(s_j) + ph(v) = (k+p)h[M(s_1, \ldots, v_p)].$$

Since the mapping $s \to h(s)$ is one-to-one, (3. 14) follows.

A subspace of R^m is said to be *principal* if it is obtained by equating k of the coordinates to zero, say $x_j = 0$ for $j = n_1, n_2, \ldots, n_k$ where the n's are fixed.

Lemma 6. *If a set V of vectors belongs to a principal linear subspace of R^m, then $C^0[V]$ belongs to the same subspace.*

Proof by inspection.

Repeated averaging with one entry fixed leads to this entry:

Lemma 7. *Let x and y be vectors in R^m and form the sequence*

(3. 15) $$x_n = M(x_{n-1}, y), \qquad n = 2, 3, \ldots, x_1 = x.$$

Then $\lim x_n = x_0$ *exists and* $x_0 = y$.

Proof. Let z_{nj} denote the j^{th} coordinate of $x_n - y$. Then for fixed j the sequence $\{z_{nj}\}$ is monotone, all the members of the sequence have the same sign namely that of the first term. The sequence is strictly increasing, zero, or strictly decreasing according as z_{1j} is negative, zero, or positive. All this follows from the definition of $M(s, t)$. Hence $\lim (x_n - y) = x_0 - y$ exists and by the continuity of M, we have $x_0 = M(x_0, y)$ so that $x_0 = y$ as asserted.

Let X and Y be subsets of G and set

(3. 16) $$M(X, Y) = \{u; u = M(s, t); s \in X, t \in Y\}.$$

If $Y = X$, then $X \subset M(X, X)$ and the inclusion is proper unless X reduces to a single vector.

4. Boundedness and singularities. For the case of the general system (1. 1) and (1. 2) we need some notions which become self-evident in the special case where B is the arithmetic mean.

The convex set G is bounded by a convex hypersurface $K = \vartheta G$. If G is not bounded, then part or all of ϑG may be infinitary. It is understood and admitted that some boundary elements may have coordinates equal to $+\infty$ or $-\infty$.

If $\{s_n\} \subset G$ and converges to an element of ϑG, we call $\{s_n\}$ a *boundary sequence*.

M is *boundary preserving* if for all $t \in G$ the sequence $\{M(s_n, t)\}$ is a boundary sequence whenever $\{s_n\}$ has this property.

As an illustration take $m=1$ and

$M(s,t)$	G	∂G
$\frac{1}{2}(s+t)$	$(-\infty, +\infty)$	$s=-\infty, +\infty;$
$\arctan[\frac{1}{2}(\tan s+\tan t)]$	$(-\frac{1}{2}\pi, \frac{1}{2}\pi)$	$s=-\frac{1}{2}\pi, +\frac{1}{2}\pi;$
$\arcsin[\frac{1}{2}(\sin s+\sin t)]$	$(-\frac{1}{2}\pi, \frac{1}{2}\pi)$	$s=-\frac{1}{2}\pi, +\frac{1}{2}\pi.$

Here the first two means are boundary preserving but the third is not.

For the special case (1. 4) local boundedness implies global boundedness. In the general case affinity for the boundary plays a similar role. This leads to the notion of a singular set.

A point $s=s_0 \in D$ is *singular* with respect to a solution $s \to f(s)$ of (1. 1) iff there exists a sequence $\{s_n\} \in D$ converging to s_0 such that $\{f(s_n)\}$ is a boundary sequence with respect to ϑE. *The set of all singular points is the singular set of f.*

5. Intern transformations. The equations (1. 1) and (1. 2) are special cases of functional equations of the form

$$f[F(x,y)] = H[f(x),(y),x,y]. \tag{5. 1}$$

Here F and H are given and f is to be found. The study of this class was initiated by J. ACZÉL [1] in 1964 for x and y real variables. F is assumed to be an *intern transformation* in the sense that

$$x<y \text{ implies } x<F(x,y)<y. \tag{5. 2}$$

Further F is continuous and H is assumed to be injective with respect to either the first or the second argument. Under these assumptions Aczél could prove that the equation has at most one continuous solution satisfying a given two-point condition

$$f(x_0)=y_0, \quad f(x_1)=y_1. \tag{5. 3}$$

Note that this is a uniqueness theorem, nothing is said about the existence of any solution.

Extensions to R^m have been proved by J. B. MILLER [3] and C. T. NG [4, 5]. Here there are two concepts to be generalized: (i) the internity and (ii) the initial conditions. *Cellular internity* in the sense that

$$F(x,y) \subset C^0(x;y) \tag{5. 4}$$

turns out to be a suitable generalization of the first notion. As to the initial conditions it is fairly clear that $m+1$ conditions

$$f(x_j)=y_j, \qquad j=0,1,2,\dots,m, \tag{5. 5}$$

are required, but it is not obvious that the x's are subject to restrictions which are satisfied if the vectors $x_k - x_0$, $k=1, 2, \ldots, m$ are linearly independent. We note that the vertices of Π are admissible choices for the x's. Assuming further F is continuous and that H is injective with respect to the first or the second argument, Ng [5] could prove that equation (5. 1) has at most one continuous solution satisfying an admissible $(m+1)$-point condition (5. 5).

This result applies in particular to equation (1. 1) and (1. 2). For by M_{21} both A and B are continuous and cellularly intern and by M_{22} both right hand sides are injective with respect to both arguments.

The case where $A=B$ is of some interest.

Theorem 1. *The equation*

$$h[A(s,t)] = A[h(s), h(t)] \tag{5.6}$$

has a unique solution $s \to h(s)$ which is continuous and leaves the vertices of Π invariant, namely $h(s) \equiv s$.

Proof by inspection. It should be noted that the solution is independent of A. This leads to the following composition theorem.

Theorem 2. *Suppose that* (1. 1) *has the continuous solution f_0 of domain D and range E which satisfies the initial conditions* (3. 11). *Suppose that g_0 is the continuous solution of* (1. 2) *with domain E and range D, satisfying the same conditions. Then $g_0[f_0(s)]$ is defined in D and equals s identically, $f_0[g_0(a)]$ is defined in E and equals a identically.*

Proof. The existence of the composite functions is obvious. In equation (1. 2) we substitute $a=f_0(s)$, $b=f_0(t)$ and obtain

$$A\{g_0[f_0(s)], \quad g_0[f_0(t)]\} = g_0\{B[f_0(s), f_0(t)]\} = g_0\{f_0[A(s,t)]\}.$$

This implies that $s \to h(s) = g_0[f_0(s)]$ is a continuous solution of (5. 6) which leaves the vertices of Π invariant. Hence $h(s) \equiv s$ as asserted. The case of $f_0[g_0(a)]$ is handled in the same manner.

6. The dichotomy. We shall prove

Theorem 3. *If f is a solution of* (1. 1) *with domain D and range E, if B is boundary preserving with respect to E and if the singular set U of f in D is not all of D, then $U = \emptyset$.*

Proof. Since D is open and connected, it is enough to prove that U is both open and closed in D for then D is either D itself or void. The first alternative being excluded by assumption, the second must hold. Suppose $s_0 \in U$. Then there is a sequence $\{s_n\}$ converging to s_0 such that $\{f(s_n)\}$ is a boundary sequence. Let $t \in D$ be arbitrary and consider the sequence $\{A(s_n, t)\}$ which converges to $A(s_0, t)$ when $n \to \infty$. Now

$$f[A(s_n, t)] = B[f(s_n), f(t)], \qquad n = 1, 2, 3, \ldots \tag{6.1}$$

and here the right members form a boundary sequence since B is boundary preserving. It follows that $\{f[A(s_n, t)]\}$ is a boundary sequence so that $A(s_0, t) \in U$. In particular, there exists a full neighborhood of $t = s_0$ which belongs to U so that U is open in D.

On the other hand, if $\{x_n\} \subset U$ and $x_n \to x_0 \in D$, there exist sequences $\{x_{nk}\}$, $n = 1, 2, 3, \ldots$, such that each $\{f(x_{nk}) : k = 1, 2, 3, \ldots\}$ is a boundary sequence. It is then seen that the diagonal sequence $\{f(x_{nn})\}$ either is a boundary sequence or contains one and here $x_{nn} \to x_0$ so that $x_0 \in U$. Thus U is also closed and, as observed above, this implies that $U = \emptyset$.

With the appropriate changes of assumptions a similar result holds for equation (1. 2).

We note that for these equations the singular set is either void or coincides with the domain of definition of the solution. In the special case (1. 4) the alternatives are a solution everywhere unbounded or one bounded on compact sets which implies continuity. In the general case where A and B are arbitrary (boundary preserving) meanvalues, we note that if the singular set is void, then the solution is necessarily bounded on compact subsets. The proof given above uses the assumption of boundary preserving but it is not obvious that this is a necessary condition for the validity of the result. For the special case the idea of the proof is due to C. T. NG.

7. Boundedness and continuity. We shall prove

Theorem 4. *If a solution of* (1. 1) *is continuous at a single point of D then it is continuous everywhere in D.*

Proof. Suppose that $s \to f(s)$ is continuous at $s = s_0$. Then for $\|h\|$ small and t arbitrary in D,

$$f[A(s_0 + h, t)] = B[f(s_0 + h), f(t)] \to B[f(s_0), f(t)]$$

while $A(s_0 + h, t) \to A(s_0, t)$. This shows that f is continuous at the point $A(s_0, t)$ for all t in D. These points form an open set $D_1 = A(s_0, D)$, a neighborhood of $s = s_0$. If $D_1 = D$ we are through. If not, we define an expanding sequence of sets $\{D_n\}$ where in the notation of (3. 16)

$$D_n = A(D_{n-1}, D), \qquad n = 2, 3, \ldots. \tag{7.1}$$

Here f is continuous in each of the sets D_n and hence also in their union D_0 and

$$D_0 = A(D_0, D). \tag{7.2}$$

by (7. 1). If t were in D but not in D_0, then we would have $A(s, t) \in D_0$ for all s in D, in particular for $s=t$. Since $A(t, t)=t$ this contradiction shows that $D_0=D$ so that f is continuous everywhere in D.

Corollary 1. *A solution of* (1. 1) *is continuous either everywhere in D or nowhere.*

The same argument applies to equation (1. 2).

In the construction of solutions given below the interpolation process leads to a solution which is not defined in all of D but merely in a dense subset T of Π. It is important for us to note that such a solution already has continuity properties provided T satisfies certain conditions. The proof given for Theorem 4 leads to

Corollary 2. *Suppose that a solution f of* (1. 1) *is defined in a subset T of D with the following properties* (i) *T is dense in Π and* (ii) *$A(T, T) \subset T$. Suppose that f is continuous at a single point s_0 of T for approach to s_0 in T, then f is continuous everywhere in T. In the case of equation* (1. 2) *replace* (ii) *by* (ii') *$B(T, T) \subset T$.*

Here $M(T, T) \subset T$ means that

$$\{x: x = M(s, t), s \in T, t \in T\} \subset T. \tag{7.3}$$

Proof. The argument used above carries over if s_0, s_0+h and t are confined to T which replaces D throughout so that

$$T_1 = A(s_0, T), \quad T_n = A(T_{n-1}, T), \quad T_0 = \bigcup_n T_n. \tag{7.4}$$

Theorem 5. *A solution of* (1. 1) *or of* (1. 2) *which is zero at the origin is continuous there and hence continuous everywhere in D. The same conclusion with D replaced by T holds if the domain of definition is a set T of the type defined above.*

Proof. Take, for instance, (1. 1) with $s \to f(s)$ as the solution with D as domain of definition and $f(0)=0$. Then for s in D we have $f[A(0, s)]=B[f(0), f(s)]=B[0, f(s)]$. We introduce two sequences of vectors $\{s_n\}$ and $\{t_n\}$ where

$$s_{n+1} = A(0, s_n), \quad s_1 = s, \quad t_{n+1} = f(s_n). \tag{7.5}$$

Here $\lim s_n=0$ by Lemma 7. Since $t_{n+1}=B(0, t_n)$, a second appeal to Lemma 7 gives $\lim t_n=0$, so that $\lim f(s_n)=0=f(0)$. This holds for every s in D so that $f(s)$ can approach no other limit than 0 as $s \to 0$. Hence f is continuous at $s=0$ and by Theorem 4 this means continuity everywhere in D. The modifications of the proof that become necessary if D is replaced by T are obvious.

Equation (1. 2) is handled in the same manner.

The arithmetic mean $\frac{1}{2}(s+t)$ is not merely a special case of the means considered here but in a certain sense it is asymptotic to the general mean. For in important cases the right member of (3. 6) can be lowered to the second power so that

$$M(s,t) = \tfrac{1}{2}(s+t) + O(\|s-t\|^2) \tag{7. 6}$$

as $t \to s$. See further below, Theorem 12.

Under these circumstances it makes sense to examine the consequences of imposing temporarily a further restriction on M, namely

M_5. There exists a constant k, $\frac{1}{2} \leqq k < 1$, such that

$$\|M(s,t) - s\| \leqq k \|s-t\|, \qquad \forall s, t \in G. \tag{7. 7}$$

It should be noted that a value of $k < \frac{1}{2}$ is not admissible for this would give $\|s-t\| \leqq 2k\|s-t\|$ which can hold only for $t=s$ if $2k<1$. On the other hand, $k=1$ is no restriction by (3. 8).

We have now

Theorem 6. *If the meanvalue B satisfies* (7. 7), *then a locally bounded solution of* (1. 1) *is locally continuous.*

Proof. Suppose that in some neighborhood of $s=s_0$ we have $\|f(s)\| \leqq F$ and set

$$\limsup_{\|h\| \to 0} \|f(s_0+h) - f(s_0)\| = \delta(s_0). \tag{7. 8}$$

Then $f[A(s_0+h, s_0)] - f(s_0) = B[f(s_0+h), f(s_0)] - f(s_0)$. By virtue of M_5 this gives

$$\|f[A(s_0+h, s_0)] - f(s_0)\| \leqq k\|f(s_0+h) - f(s_0)\|. \tag{7. 9}$$

Here we let $\|h\| \to 0$. The superior limit of the right hand side is $k\delta(s_0)$ which is non-negative and at most equal to $2kF$. In the left member $h \to A(s_0+h, s_0)$ maps a sphere with center at $h=0$ onto a full neighborhood of $s=s_0$. It follows that the superior limit of the left member is $\delta(s_0)$ when $\|h\| \to 0$ so we have $\delta(s_0) \leqq k\delta(s_0)$. This implies $\delta(s_0)=0$ and makes f continuous at $s=s_0$ and hence everywhere.

The same argument applies to (1. 2) if A satisfies M_5. Moreover, we can replace D by a set T with but little change in the argument.

8. Primary mappings. So far we have operated under the assumption that solutions of our equations do exist. This will now be proved. More precisely, we shall prove that solutions satisfying the initial conditions (3. 11) can be constructed in the basic pyramid Π. These are the so called *fundamental solutions* and since they are continuous at the origin they are continuous wherever they are defined. We concentrate on equation (1. 1) but the same method applies to (1. 2). Actually we construct solutions of the auxiliary equations

$$h[\tfrac{1}{2}(a+b)] = A[h(a), h(b)], \tag{8. 1}$$

$$k[\tfrac{1}{2}(a+b)] = B[k(a), k(b)], \tag{8. 2}$$

with the aid of which we obtain the solutions of (1. 1) and (1. 2). These equations involve the arithmetic mean. We set

$$C(a, b) = \tfrac{1}{2}(a+b). \tag{8. 3}$$

Let R be the set of points in Π with dyadic rational coordinates. It is clear that R is dense in Π and we have also

$$R \supset C(R, R) \tag{8. 4}$$

so that R is a set T in the sense defined above. Next we construct two sets S and Y by applying the operations A and B repeatedly to R. It simplifies matters to construct these sets step by step. Let $R_0 = S_0 = Y_0$ be the set of $(m+1)$ vertices of Π and define the sets $\{R_n\}$, $\{S_n\}$ and $\{Y_n\}$ recursively as follows

$$R_n = C(R_{n-1}, R_{n-1}), \quad S_n = A(S_{n-1}, S_{n-1}), \quad Y_n = B(Y_{n-1}, Y_{n-1}). \tag{8. 5}$$

These are sequences of expanding bounded sets and clearly tend to limits which are the sets R, S, T.

None of these sets is closed but we shall prove

Lemma 8. *The closures of the sets R, S, Y are connected, even simply connected.*

Proof. For R this is trivial. We shall give the argument for S, the same proof applies to Y. If a and b are two points of S, we set $T_0 = a \cup b$ and $T_n = A(T_{n-1}, T_{n-1})$ and $T = \bigcup_0^\infty T_n$. Then $\bar{T}$ is a continuous arc joining a with b and lies in $\bar{S}$. Since S is closed under the operation A, it is clear that $T \subset S$. By Lemma 7 every point of T is a limit point of T and since A is uniformly continuous, it follows that $\bar{T}$ is a continuous arc.

To prove that $\bar{S}$ is simply-connected we have to show that the complement of $\bar{S}$ is connected. If this is not so, then there is a point z_0 in the complement of $\bar{S}$ that cannot be joined to a point in the infinitary component of the complement of $\bar{S}$ by a Jordan arc not having points in common with $\bar{S}$. We can then "box in" z_0, that is we can find a parallelopiped π such that: (i) The faces of π are planes $x_j = \alpha_j$, $x_j = \beta_j$, $\alpha_j < \beta_j$, $j = 1, 2, \ldots, m$. (ii) There is no point of $\bar{S}$ in the interior of π. (iii) For each j there is a point $P_{j1} \in \bar{S}$ on the face $x_j = \alpha_j$ and a point $P_{j2} \in \bar{S}$ on the face

$x_j = \beta_j$. Since these points may fall on edges or vertices of π the total number of distinct points does not have to be $2m$ in number, but there are at least two. To see that this extreme case can happen, note that each of the endpoints of a diagonal of π lies on m distinct faces and that these faces have nothing in common with the faces associated with the other endpoint. If this case should be present, we obtain a contradiction right away for the A-mean of the endpoints is an interior point of π and belongs to $\bar{S}$ if the endpoints do. In the general case we may have to perform as many as $(m+1)$ A-operations before a contradiction results. To see this we form $P_j = A(P_{j1}, P_{j2})$. This is a point of $\bar{S}$, it may conceivably fall on an edge or a face of π but in any case its j^{th} coordinate belongs to the open interval (α_j, β_j) by virtue of the properties of $C^0(x, y)$. If one of the points P_j belongs to the interior of π, we are through. If not, one more averaging will lead to the desired contradiction. We form $P = A(P_1, P_2, \ldots, P_m)$. Again this is a point of $\bar{S}$ but now for each j the jth coordinate belongs to the open interval (α_j, β_j) and this forces P to be an interior point of π thus violating assumption (ii). This contradiction shows that $\bar{S}$ is simply-connected.

A closer examination of the generation of the sets R, S, Y is now in order. An element of R is obtained by applying the arithmetic means, the operation C, to some of the elements of the set R_0; i.e., the vertices of Π, say

(8.6) $$k_0 \text{ copies of } 0,\ k_1 \text{ copies of } u_1, \ldots, k_m \text{ copies of } u_m,$$

where $\sum_{j=0}^{m} k_j = 2^n$ for some positive integer n. Denote this aggregate of 2^n vectors by V. Then

(8.7) $$a = C(V).$$

To this element of R correspond vectors of S and Y

(8.8) $$s = A(V) = p(a), \quad y = B(V) = q(a).$$

Our first object is to study these mappings of R into S and Y.

Lemma 9. *The mappings $R \xrightarrow{p} S$ and $R \xrightarrow{q} Y$ are one-to-one.*

Proof. Take two distinct points a and b of R and suppose that

(8.10) $$a = C(V_1), \quad b = C(V_2)$$

in the notation of (8.6) and (8.7). Since we can always enlarge our vector sets using Lemma 1 we may assume that V_1 and V_2 both contain 2^n vectors. The sets V_1 and V_2 are distinct, but may have elements in common. Let V_0 be the set of common vectors; if u_j occurs μ_j times in V_1 and ν_j times in V_2 then it occurs $\min(\mu_j, \nu_j)$ in V_0. Set

$$V_1 = V_0 \cup U_1, \quad V_2 = V_0 \cup U_2. \tag{8.11}$$

Then U_1 and U_2 have the same number of elements but have no vectors in common. Moreover, the linear manifolds spanned by U_1 and U_2 have only the zero element in common. In any case Lemma 6 shows that $C^0[U_1] \cap C^0[U_2] = \emptyset$ and this implies that $A(U_1) \neq A(U_2)$. By Lemma 2

$$p(a) = A(V_1) = A(V_0 \cup U_1) = A[A(V_0), A(U_1)],$$

$$p(b) = A(V_2) = A(V_0 \cup U_2) = A[A(V_0), A(U_2)],$$

and by (3. 13) $p(a) \neq p(b)$. In the same manner it is shown that $q(a) \neq q(b)$.

Theorem 7. *The fundamental solution of* (8. 1) *is* $h(a) = p(a)$ *and the fundamental solution of* (8. 2) *is* $k(a) = q(a)$.

Proof. Consider $A[p(a), p(b)]$ with $p(a) = A(V_1)$, $p(b) = A(V_2)$. Then $A[p(a), p(b)] = A[A(V_1), A(V_2)] = A(V_1 \cup V_2)$. Without loss of generality we may suppose that V_1 and V_2 contain the same number of elements, namely 2^n. Then $V_1 \cup V_2$ contains 2^{n+1} vectors and $C(V_1 \cup V_2) = \frac{1}{2}(a+b)$. Hence $A(V_1 \cup V_2) = p[\frac{1}{2}(a+b)]$. This proves that p satisfies (8. 1) for $a, b \in R$. Equation (8. 2) is handled in the same manner.

Lemma 10. *The mappings* $a \to p(a)$ *and* $a \to q(a)$ *are continuous on* R, *uniformly in* $(1-2^{-n})R$ *for any* n.

Proof. Since R is a set of type T and since $p(0) = q(0) = 0$, Theorem 5, gives continuity in R. Only the uniformity remains to be proved. Suppose that b and h are in R. Then with M and g as generic notations

$$M[g(b), g(h)] = g[\tfrac{1}{2}(b+h)].$$

Letting $h \to 0$ in R, we get continuity at $\frac{1}{2}b$ uniformly with respect to b in R. This gives uniform continuity in $(1-2^{-n})R$ for $n=1$ and the general case follows by induction.

9. First extension. Our functions p and q are defined in R which is dense in Π. The next step is to extend to all of Π.

Theorem 8. *The mappings* $a \to p(a)$, $a \to q(a)$ *can be extented as continuous mappings to all of* Π.

This follows from the fact that R is dense in Π and the primary mappings are uniformly continuous on any set $(1-2^{-n})R$.

Lemma 11. *The extented mappings are one-to-one.*

Proof. The argument is essentially that used for Lemma 8. Consider two distinct points a and b in Π. Here

$$a = \sum_{j=1}^{m} \alpha_j u_j, \quad \sum \alpha_j \leqq 1, \quad b = \sum_{j=1}^{m} \beta_j u_j, \quad \sum \beta_j \leqq 1, \tag{9.1}$$

and the α's and β's are real, non-negative. Set

$$c = \inf(a, b), \tag{9.2}$$

$$a = c + a_1, \quad b = c + b_1. \tag{9.3}$$

Here the unit vectors u_j which enter in a_1 are distinct from those of b_1. Either a_1 or b_1 may be zero but not both. If this should happen we have $g(a_1) \neq g(b_1)$. If both are positive vectors, they belong to principal linear subspaces having only the zero vector in common. Again $g(a_1) \neq g(b_1)$. Now

$$M[g(c), g(a_1)] = g[\tfrac{1}{2}(c + a_1)] = g(\tfrac{1}{2}a), \quad M[g(c), g(b_1)] = g[\tfrac{1}{2}(c + b_1)] = g(\tfrac{1}{2}b).$$

Here the first members are distinct by (3. 13), hence also the last. But if $g(a) = g(b)$, then $g(\frac{1}{2}a) = g(\frac{1}{2}b)$ since we would have $g(\frac{1}{2}a) = M[g(a), 0] = M[g(b), 0] = g(\frac{1}{2}b)$ which is a contradiction.

We can now construct fundamental solutions of the original equations (1. 1) and (1. 2). Since $a \to p(a)$ is one-to-one on Π there exists a unique inverse

$$a = P(s), \quad s = p(a), \quad a \in \Pi, \tag{9.4}$$

which is continuous and one-to-one.

Theorem 9. *The fundamental solution of* (1. 1) *is given by the mapping*

$$s \to f(s) = q[P(s)]. \tag{9.5}$$

Proof. Let $s = p(a)$, $t = p(b)$ where a and b are in Π. Then

$$A(s, t) = A[p(a), p(b)] = p[\tfrac{1}{2}(a + b)]$$

so that

$$f[A(s, t)] = f\{p[\tfrac{1}{2}(a + b)]\} = q[\tfrac{1}{2}(a + b)]$$

by the definition of f. On the other hand,

$$B[f(s), f(t)] = B\{q[P(s)], q[P(t)]\} = B[q(a), q(b)] = q[\tfrac{1}{2}(a + b)].$$

This shows that f is a solution of (1. 1) defined on the image of Π under the mapping $a \xrightarrow{p} s$. Since both p and q leave the vertices of Π invariant, the same is true for f so that f is the fundamental solution of (1. 1). It is obviously continuous since P and q are continuous.

In the same manner it is shown, that

$$p[Q(u)] = q(u), \quad Q(u) = a, \quad u = q(a)$$

is the fundamental solution of (1. 2).

10. Second extensions. We shall indicate a method of extending our functions to the outside of Π. We base the argument on the functional equations and are guided by a Principle of Permanence of Functional Equations. The letters G and M are used in the generic sense as above. We start by giving an elaborate interpretation of Postulate M_{22}.

Theorem 10. *For a given $z \in G$ there exists a positive number σ and a mapping $x \to H(x, z)$ with the following properties.*

(1) *H is defined and continuous for $\|x-z\| < \sigma$ and x its values lie in G.*

(2) *H is injective with respect to x.*

(3) *$H(z, z)=z$.*

(4) *Por $\|x-z\| < \sigma$*

$$M[x, H(x, z)] \equiv z. \tag{10.1}$$

(5) *H is unique.*

Proof. By M_{22} the mapping $t \to M(s, t)$ is open, injective and continuous, uniformly on compact subsets of G. Suppose that G_0 is such a set and $z \in G_0$. Let r be a fixed number $0<r<\frac{1}{2}d[G_0, \partial G]$. Then there exists a positive number $\varrho=\varrho(r)$ such that for each $s \in G_0$ the set

$$E_s = \{u; u = M(s, t), \|t-s\| < r\} \tag{10.2}$$

contains an open sphere with center at $u=s$ and radius at least $\varrho(r)$. Take now the point z and choose an x at a distance $\leqq \frac{1}{2}\varrho(r)$ from z. Then the set E_x contains the open sphere $\|u-x\| < \varrho(r)$ and, in particular, the point z. Hence there is a vector $t=y$ with $\|y-x\| < r$ such that

$$M(x, y) = z. \tag{10.3}$$

Moreover, since the mapping is injective, there is one and only one such vector

$$y \doteq H(x, z). \tag{10.4}$$

This is H and for σ we can take $\frac{1}{2}\varrho(r)$. The continuity of M implies the continuity of H and H is injective in x because M has this property. This completes the proof.

We come now to the applications of this theorem to the extension problem. The results will be stated as lemmas. We consider first radial extension.

Lemma 12. *Let $a \to g(s)$ be a solution of*

$$M[g(a), g(b)] = g[\tfrac{1}{2}(a+b)] \tag{10.5}$$

which is defined, continuous and with values in G for a on a ray, $a=\alpha a_0$, $0\leqq\alpha\leqq 1$, $a_0\neq 0$. Then there exists a τ, $1<\tau<\infty$ such that g can be defined for $1<\alpha\leqq\tau$ with the same properties.

Proof. In Theorem 10 we take

$$(10.6)\qquad g[(1-\eta)a_0]=x,\quad g(a_0)=z,\quad g[(1+\eta)a_0)]=y.$$

Here η is a small positive number, so small that $\|x-z\|<\frac{1}{2}\varrho(r)$ with r referring to a compact set containing such parts of the ray which are needed for the argument. Theorem 10 then asserts the existence of a unique $y=H(x,z)$ and we can satisfy (10. 5) by defining

$$(10.6)\qquad g[(1+\eta)a_0]=H(x,z)=H\{g[(1-\eta)a_0],\, g(a_0)\}.$$

We can take $\tau=1+\eta$ for any admissible η.

This lemma enables us to cross the "base" of the pyramid Π, i.e. the plane

$$x_1+x_2+\cdots+x_m=1.$$

In particular, g will be defined along the positive axes some distance beyond the vertices of Π.

So far g has been defined only for non-negative vectors. The extension to negative vectors is again based on Theorem 10.

Lemma 13. *If the origin is an interior point of G and if $g(0)=0$ and a_0 is a vector such that $g(a)$ is defined for $a=\alpha a_0$, $0\leqq\alpha\leqq 1$, $a_0\neq 0$, then there exists a $\tau>0$ such that*

$$(10.7)\qquad g(-\alpha a_0)=H[g(\alpha a_0),0]$$

for $0\leqq\alpha\leqq\tau$.

Proof by inspection. In particular, g is definable on the negative axes for sufficiently small values of $\|a\|$.

Lemma 14. *The domain of definition of g is a convex subset of R^m.*

Proof. Equation (10. 5) shows that g is defined at the point $\frac{1}{2}(a+b)$, if it is known at a and at b and if $g(a)$ and $g(b)$ belong to G where the meanvalue M is defined.

Lemma 15. *If g satisfies* (10. 4) *for all a and b in a convex domain D^*, then for any n and for any choice of $a_1, a_2, \ldots, a_n$ in D^* we have*

$$(10.8)\qquad M[g(a_1), g(a_2), \ldots, g(a_n)]=g\left[\frac{1}{n}(a_1+a_2+\cdots+a_n)\right].$$

Proof. The lemma is true for $n=2^k$ by induction on k and by retrogressive induction (following the precepts of Cauchy) from n to $n-1$.

Lemma 16. *The mapping $a \to g(a)$ is one-to-one.*

Proof. The argument given for Lemma 11 applies to the present case.

Applying these results to our functional equations (1. 1), (1. 2), (8. 1) and (8. 2) we are led to the following existence theorem.

Theorem 11. *Let the meanvalues A and B be defined in the convex domains D and E respectively, and satisfy the M-postulates. Let $\Pi \in D \cap E$.*

The fundamental solution $p(a)$ of (8. 1) *is defined and continuous in a convex domain C_1, $\Pi \subsetneq C_1$. The range R_1 of p is contained in D. The mapping $a \to p(a)$ is one-to-one.*

For the fundamental solution $q(a)$ of (8. 2) *replace C_1, R_1, D by C_2, R_2, E.*

The fundamental solution of (1. 1) *is defined by*

(10. 9) $$f(s) = q[P(s)], \quad s = p(a), \quad a = P(s)$$

for $s \in S = p(C_1 \cap C_2) \subset R_1$ where f is continuous and one-to-one.

The fundamental solution g of (1. 2) *is defined by*

(10. 10) $$g(u) = p[Q(u)], \quad Q(u) = q, \quad u = q(a)$$

for $u \in U = q(C_1 \cap C_2) \subset R_2$ where g is continuous and one-to-one.

If $C_1 = C_2$, then f is defined in R_1 and g in R_2. It is reasonable to expect that R_1 and R_2 are also convex but this question must be left open.

11. Further comments. We have restricted ourselves to fundamental solutions. Initial conditions of the type

(11. 1) $$g(0) = y_0, \quad g(u_j) = y_j, \qquad j = 1, 2, \dots, m,$$

or more generally

(11. 2) $$g(x_k) = y_k, \qquad k = 0, 1, 2, \dots, m,$$

present a new and more difficult problem. The methods used above may or may not be effective in this case. Moreover, normally there is no simple relation between different solutions so information on the fundamental solution has little bearing on the behavior of other solutions.

This is totally different from the case

(11. 3) $$f[A(s, y)] = \tfrac{1}{2}[f(x) + f(y)]$$

where the general solution f and the fundamental solution f_0 satisfy

(11. 4) $$f(x) = \mathscr{C} f_0(x) + v_0.$$

Here $\mathscr{C}$ is an arbitrary m by m matrix and v_0 is an arbitrary vector. It is of course the linear character of (11. 3) with respect to f that accounts for the difference. Let us note in passing

Lemma 17. *The mapping $x \to f(x)$ is one-to-one iff $\mathscr{C}$ is non-singular.*

Proof. If $\mathscr{C}$ is non-singular then $f(x_1)-f(x_2) = \mathscr{C}[f_0(x_1)-f_0(x_2)]$. The quantity in square brackets is different from 0 if $x_1 \neq x_2$ so the same must hold for the left member. On the other hand, if $\mathscr{C}$ is singular we can find a vector w_0 such that $w_0 \neq 0$ and $\mathscr{C}w_0=0$. Here $\|w_0\|$ is at our disposal and may be chosen so small that there is an $x_0 \neq 0$ but small so that $f_0(x_0)=w_0$. Then $f(x_0) = \mathscr{C}f(x_0)+v_0 = \mathscr{C}w_0+v_0 = v_0 = f(0)$, so the mapping is not one-to-one.

It was observed above, formulas (3. 6) and (7. 6), that $M(s, t)-\frac{1}{2}(s+t)$ is small and at most $O(\|s-t\|)$. This can be made more precise.

Lemma 18. *Let $m=1$ and suppose that the generating function h of Theorem M has continuous first and second order derivatives. If $s<t$ there exists an s_1, $s<s_1<t$, such that*

$$M(s, t) = \tfrac{1}{2}(s+t)+\tfrac{1}{8}\frac{h''(s_1)}{h'(s_1)}(t-s)^2. \tag{11.5}$$

Proof. Since $M(s, s)=s$ and $\frac{1}{2}(s+t) = s+\frac{1}{2}(t-s)$ it is evidently required to prove that

$$M_s(s, t)|_{t=s} = \tfrac{1}{2}, \quad M_{ss}(s, t)|_{t=s} = \tfrac{1}{4}\frac{h''(s)}{h'(s)}. \tag{11.6}$$

These relations we obtain from the functional equation (3. 10)

$$h[M(s, t)] = \tfrac{1}{2}[h(s)+h(t)].$$

Differentiation with respect to s gives

$$h'[M(s, t)]\,M_s(s, t) = \tfrac{1}{2}h'(s),$$

$$h''[M(s, t)]\,[M_s(s, t)]^2+h'[M(s, t)]\,M_{ss}(s, t) = \tfrac{1}{2}h''(s).$$

Here we put $t=s$ and solve for $M_s(s, s)$ and $M_{ss}(s, s)$, noting that $h'(s) \neq 0$ since h is strictly monotone. The result is (11. 6), and (11. 5) is Taylor's theorem with remainder.

This is elementary and so is the extension to $m>1$ but the latter requires a number of devices and a much more thorough use of the functional equation.

Theorem 12. *If the generating function h of Theorem M has continuous first and second order partial derivatives, then*

$$M(s, t) = \tfrac{1}{2}(s+t)+O(\|s-t\|^2). \tag{11.7}$$

Proof. Let $z=(z_1, z_2, \ldots, z_m) \in R^m$, $s=(s_1, s_2, \ldots, s_m) \in G$ and let α be real and so small that $s+\alpha z \in G$. Consider $M(s, s+\alpha z)$. We have

$$h[M(s, s+\alpha z)] = \tfrac{1}{2}[h(x)+h(x+\alpha z)]. \tag{11.8}$$

The assumption on h implies that the right member may be differentiated with respect α and hence also the left member so that $M(s, s+\alpha z)$ is differentiable. Set

$$h = (h_1, h_2, \ldots, h_m), \quad M = (M_1, M_2, \ldots, M_m).$$

Each h_j is a real-valued function of the m real arguments $M_1, M_2, \ldots, M_m$ and is differentiable with respect to each of them. Let $h_{j,k}$ denote the partial of h_j with respect to the k^{th} argument with similar notation for other vector functions. For a fixed j we equate the j^{th} components of the two sides in (11. 7) and differentiate once with respect to α to obtain

$$\sum_{k=1}^{m} h_{j,k}(*) \sum_{p=1}^{m} M_{k,p}(*) z_p = \tfrac{1}{2} \sum_{k=1}^{m} h_{j,k}(\cdot) z_k \tag{11.9}$$

where

$$(\cdot) = (s_1+\alpha z_1, s_2+\alpha z_2, \ldots, s_m+\alpha z_m), \quad (*) = \big(M_1(\cdot), M_2(\cdot), \ldots, M_m(\cdot)\big).$$

We now set $\alpha=0$ in (11. 9). Both $(\cdot)$ and $(*)$ collapse and become $(s_1, s_2, \ldots, s_m)=s$ so that (11. 9) becomes

$$\sum_{k=1}^{m} h_{j,k}(s) \sum_{p=1}^{m} M_{k,p}(s) z_p = \tfrac{1}{2} \sum_{k=1}^{m} h_{j,k}(s) z_k. \tag{11.10}$$

Since z is arbitrary, (11. 10) must hold identically in the components $z_1, z_2, \ldots, z_m$. This gives a system of m equations

$$\sum_{k=1}^{m} h_{j,k}(s)[M_{k,p}(s) - \tfrac{1}{2}\delta_{kp}] = 0, \qquad p = 1, 2, \ldots, m, \tag{11.11}$$

which, regarded as a system of equations for the derivatives of the components of h, certainly has non-trivial solutions. Hence the matrix

$$\mathscr{J}(s) - \tfrac{1}{2}\mathscr{E}_m \tag{11.12}$$

is singular. Here $\mathscr{E}_m$ is the unit matrix in R^m and $\mathscr{J}(s)$ is a Jacobian

$$\mathscr{J}(s) = \big(M_{k,p}(s)\big). \tag{11.13}$$

We note that $\mathscr{J}(s)$ has the characteristic value $\frac{1}{2}$ for all values of s. Actually this is the only characteristic value and we shall prove the stronger result

$$\mathscr{J}(s) \equiv \tfrac{1}{2}\mathscr{E}_m. \tag{11.14}$$

To this end we note that (11. 8), regarded as a functional equation for h when M is given, has infinitely many solutions which, as observed above, are of the form (11. 4)

$$h(s) = \mathscr{C}h_0(s) + v_0.$$

Here h_0 is the fundamental solution, $\mathscr{C}$ is an arbitrary m by m constant matrix and v_0 is a vector with constant components. We can use this freedom of choice to normalize equations (11. 11) for a particular but arbitrary value of s, $s = s_0 \in G$. The set of m systems of m equations will obviously simplify very much, in fact become trivial, if we can determine the matrix $\mathscr{C}$ so that

(11. 15) $$h_{j,k}(s_0) = \delta_{j,k}, \qquad j, k = 1, 2, \ldots, m$$

To attain this, note that the matrix

(11. 16) $$\mathscr{H}^0(s) = (h^0_{j,k}(s))$$

of the first order derivatives of the components of the fundamental solution h_0 is necessarily non-singular since the mapping $s \to h_0(s)$ is one-to-one. It follows that we can solve equations of the form

(11. 17) $$\sum_{p=1}^{m} c_{jp} h^0_{p,k}(s_0) = h_{j,k}(s_0), \qquad j, k = 1, 2, \ldots, m$$

for the c's. We have m systems corresponding to a fixed value of k, all systems with the same non-vanishing determinant, namely that of the matrix $\mathscr{H}^0(s_0)$. In particular, this is possible if the right hand sides are given by (11. 15). This determines the matrix $\mathscr{C}$ and the normalizing transformations at $s = s_0$. The equations (11. 11) now give

(11. 16) $$M_{k,p}(s_0) = \tfrac{1}{2}\delta_{kp}, \qquad k, p = 1, 2, \ldots, m$$

and (11. 14) holds for $s = s_0$. Since s_0 is arbitrary in G, (11. 14) must hold identically in s.

This gives

$$M(s, s+\alpha z)|_{\alpha=0} = \left(\sum_{p=1}^{m} M_{1,p}(s) z_p, \ldots, \sum_{p=1}^{m} M_{m,p}(s) z_p \right) = \tfrac{1}{2}(z_1, z_2, \ldots, z_m) = \tfrac{1}{2} z.$$

We take $z = t - s$ and note again that $s + \frac{1}{2}(t-s) = \frac{1}{2}(s+t)$ which is the first term in the right member of (11. 7).

To get the remainder we can compute

$$\frac{\partial^2}{\partial \alpha^2} M(s, s+\alpha z)|_{\alpha=0} = \delta^2 M(s; z).$$

We do not insist on the exact expression for the second variation, it is clear that it is $O(\|z\|^2)$ uniformly in s on compact subsets of G.

Corollary. *Under the assumptions of Theorem* 12 *we have, for any n and for any choice of n vectors in G*

$$M(s_1, s_2, \ldots, s_m) = \frac{1}{n}(s_1 + s_2 + \cdots + s_m) + R \tag{11.19}$$

where $\|R\|$ *is dominated by a constant multiple of*

$$\sum_{1 \leq j < k \leq n} (\|s_j - s_k\|^2)$$

uniformly on compact subsets of G.

Proof. For $n = 2^k$ use induction on k. Complete by retrogressive induction from n to $n-1$. This gives the leading term. Note that the remainder must be a symmetric function of the s-vectors and should vanish when they are all equal.

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